1 Analysis of Work

Define an indicator random variable $X_k$ as 1 if $k$th rank element is the pivot when partitioning the array. We can see that the recurrence is

$$W(n) = \Theta(n) + \sum_{k=0}^{n-1} X_k(W(k) + W(n - k - 1))$$

assuming that partition has a span of $O(n)$.

We can now take expectation

$$E[W(n)] = \Theta(n) + E\left[\sum_{k=0}^{n-1} X_k(W(k) + W(n - k - 1))\right]$$

$$= \Theta(n) + \sum_{k=0}^{n-1} E[X_k(W(k) + W(n - k - 1))]$$

$$= \Theta(n) + \sum_{k=0}^{n-1} E[X_k]E[W(k) + W(n - k - 1)]$$

$$\leq \Theta(n) + \frac{2}{n} \sum_{k=0}^{n-1} E[W(k)]$$

Now you can guess the solution of this recurrence is $O(n \lg n)$ and try to prove it using induction. Say $E[W(n)] = c_1 n \lg n + c_0$.

**Base case:** When $n = 1$, it trivially works as long as $c_0$ is large enough.

**Inductive step:** Say that it is true for all $k < n$. We can say

$$E[W(n)] \leq cn + \frac{2}{n} \sum_{k=0}^{n-1} E[W(k)]$$

$$\leq cn + \frac{2}{n} \sum_{k=0}^{n-1} (c_1 k \lg k + c_0)$$

$$\leq cn + 2c_0 + c_1 \frac{2}{n} \sum_{k=0}^{n-1} k \lg k$$
Now we must compute $\sum_{k=0}^{n-1} k \lg k$. There are various ways of doing this, but we can approximate using an integral and get something like $n^2 \lg n/2 - n^2/8$. If you substitute it back into the original equation, you get the right answer as long as $c_1$ is large enough.

2 Analysis of Span

The recurrence is $S(n) = \max\{S(k), S(n - k - 1)\} + \Theta(\lg n)$ if rank $k$ element is the pivot element.

Define an indicator random variable $X_k$ as 1 if $k$th rank element is the pivot when partitioning the array. Therefore, the recurrence is

$$S(n) = \Theta(\lg n) + \sum_{k=0}^{n-1} X_k \max\{S(k) + S(n - k - 1)\}$$

assuming that scan has a span of $O(\lg n)$.

**Exercise 1** Show that $E[S(n)] = O(\lg^2 n)$. You may assume that $\sum_{k=[n/2]}^{n-1} \lg^2 k \leq n/2 \lg^2 n - n \lg n + n$. 

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